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Chromaticity of 6-bridge Graph $\theta(3, 3, b, b, c, c)$

Nor Suriya Abd Karim [1*] and Roslan Hasni [2]

[1]*Department of Mathematics, Faculty of Science and Mathematics, Universiti Pendidikan Sultan Idris, 35900 Tanjung Malim, Perak, Malaysia*

[2]*School of Informatics and Applied Mathematics, Universiti Malaysia Terengganu, 21030 Kuala Terengganu, Terengganu, Malaysia*

E-mail: yaya_kulaan@yahoo.com

*Corresponding author

ABSTRACT

For a graph G, let $P(G, \lambda)$ denote the chromatic polynomial of G. Two graphs G and H are chromatically equivalent (or simply χ-equivalent), denoted by $G \sim H$, if $P(G, \lambda) = P(H, \lambda)$. A graph G is chromatically unique (or simply χ-unique) if for any graph H such that $H \sim G$, we have $H \cong G$, that is, H is isomorphic to G. In this paper, the chromatic uniqueness of a family of 6-bridge graph is determined.

Keywords: Chromatic polynomial, chromatically unique, 6-bridge graph.

1. Introduction

All graphs considered here are finite and undirected graphs. For a graph G, let $P(G, \lambda)$ denote the chromatic polynomial of G. Two graphs G and H are chromatically equivalent (or simply χ-equivalent), denoted by $G \sim H$, if $P(G, \lambda) = P(H, \lambda)$. A graph G is chromatically unique (or simply χ-unique) if for any graph H such that $H \sim G$, we have $H \cong G$, i.e, H is isomorphic to G. For each integer $k \geq 2$, let θ_k be a multigraph with two vertices and k edges. Any subdivision of θ_k is called a multi-bridge graph or a k-bridge graph. Let $\boldsymbol{N}$ denote the set of natural numbers. We denote $\theta(a_1, a_2, \ldots, a_k)$ where $a_1, a_2, \ldots, a_k \in \mathbf{N}$ and $a_1 \leq a_2 \leq \ldots \leq a_k$ be a graph obtained by replacing the edges of θ_k by paths of length $a_1, a_2, \ldots, a_k$, respectively, as in Figure 1.

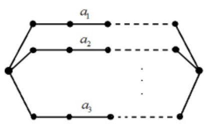

Figure 1: k-bridge graph.

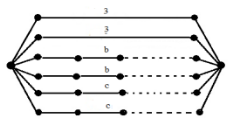

Figure 2: $\theta(3,3,b,b,c,c)$

The study on the chromaticity of k-bridge graph have been done by several researchers (see Koh and Teo (1990), (1997); Dong et al. (2005)). A 2-bridge graph is simply a cycle and it is χ-unique (Chao and Whitehead Jr. (1978)). A 3-bridge graph of the form $\theta(1, a_2, a_3)$ is called θ-graph and 3-bridge graph of the form $\theta(a_1, a_2, a_3)$ is called generalized θ-graph. These graphs are χ-unique (Chao and Whitehead Jr. (1978); Loerinc (1978)). Later on, Xu et al. (1994) and Chen et al. (1992), solved the chromaticity of 4-bridge graph. The chromaticity of 5-bridge graphs have been solved completely (Bao and Chen (1994), Li (2008), Li and Wei (2001), Khalaf (2010), Khalaf and Peng (2009a) and Khalaf et al. (2010)). Recently, Khalaf and Peng (2010), proved that 6-bridge graph $\theta(a_1, a_2, \dots, a_6)$ where $a_1, a_2, \dots, a_6$ are assumed exactly two distinct values is χ-unique. More result on the chromaticity of 6-bridge graph $\theta(a_1, a_2, \dots, a_6)$ can be obtained in some papers (Khalaf (2010)), Khalaf and Peng (2009b), (2009c)). So far, the problem of chromaticity on 6-bridge graph is still not completely solved. In this paper, we investigate the chromaticity of a family of 6-bridge graph of the form $\theta(3,3,b,b,c,c)$ where $3 \leq b \leq c$ (see Figure 2).

2. Preliminary Results

In this section, we give some known results used in this paper.

Theorem 1. For $k \geq 4$, $\theta(a_1, a_2, \dots, a_k)$ is χ-unique if $k\text{-}1 \leq a_1 \leq a_2 \leq \dots \leq a_k$ (Xu et al. (1994)).

For each positive integer h, the graph $G(h)$ is obtained from G by replacing each edge of G by path of length h, respectively and is called the h-uniform subdivision of G. Then, $\theta_k(h)$ is the h-uniform subdivision of θ_k.

Theorem 2. For $k \geq 2$, the graph $\theta_k(h)$ is χ-unique (Xu et al. (1994)).

Theorem 3. If $2 \leq a_1 \leq a_2 \leq \dots \leq a_k < a_1+a_2$, where $k \geq 3$, then the graph $\theta(a_1, a_2, \dots, a_k)$ is χ-unique (Dong et al. (2004)).

Theorem 4. For any k, a_1, a_2, …, $a_k \in \mathbf{N}$,

$$Q(\theta(a_1,a_2,...,a_k),x) = x\prod_{i=1}^{k}(x^{a_i}-1) - \prod_{i=1}^{k}(x^{a_1}-x) \qquad \text{(Dong et al. (2004)).}$$

Theorem 5. For any graphs G and H,

i) If $H \sim G$, then $Q(H,x) = Q(G,x)$,
ii) If $Q(H,x) = Q(G,x)$ and $v(H) = v(G)$, then $H \sim G$ (Dong et al. (2004)).

Theorem 6. Suppose that $\theta(a_1, a_2, \ldots, a_k) \sim \theta(b_1, b_2, \ldots, b_k)$, where $k \geq 3$, $2 \leq a_1 \leq a_2 \leq \ldots \leq a_k$ and $2 \leq b_1 \leq b_2 \leq \ldots \leq b_k$, then $a_i = b_i$ for all $i = 1, 2, \ldots, k$ (Dong et al. (2004)).

Suppose $g_e(\theta(b_1,b_2,...,b_t),C_{b_{t+1}+1},...,C_{b_k+1})$ be the collection of edge-gluing of θ-graph and cycle graphs.

Theorem 7. Let $H \sim \theta(a_1, a_2, \ldots, a_k)$ where $k \geq 3$ and $a_i \geq 2$ for all i, then one of the following is true:

i) $H \cong \theta(a_1, a_2, \ldots, a_k)$,
ii) $H \in g_e(\theta(b_1,b_2,...,b_t),C_{b_{t+1}+1},...,C_{b_k+1})$, where $3 \leq t \leq k\text{-}1$ and $b_i \geq 2$ for all $i = 1, 2, \ldots, k$ (Dong et al. (2004)).

Theorem 8. Let k, t, b_1, b_2, …, $b_k \in N$ with $3 \leq t \leq k\text{-}1$ and $b_i \geq 2$ for all $i = 1, 2, \ldots, k$. If $H \in g_e(\theta(b_1,b_2,...,b_t),C_{b_{t+1}+1},...,C_{b_k+1})$, then

$$Q(H,x) = x\prod_{i=1}^{k}(x^{b_i}-1) - \prod_{i=1}^{t}(x^{b_1}-x)\prod_{i=t+1}^{k}(x^{b_i}-1) \qquad \text{(Dong et al. (2004)).}$$

Theorem 9. If $G \sim H$, then

i) $v(G) = v(H)$,
ii) $e(G) = e(H)$,
iii) $g(G) = g(H)$,
iv) G and H have the same number of shortest cycle,

where $v(G)$, $v(H)$, $e(G)$, $e(H)$, $g(G)$ and $g(H)$ denote the number of vertices, the number of edges and the girth of G and H, respectively (Koh and Teo (1990)).

Theorem 10. A 6-bridge graph $\theta(a_1, a_2, \ldots, a_6)$ is χ-unique if the positive integers $a_1, a_2, \ldots, a_6$ assume exactly two distinct values (Khalaf and Peng (2010)).

3. Main Result

In this section, we present our main result.

Theorem 11. A 6-bridge graph of the form $\theta(3,3,b,b,c,c)$ where $3 \le b \le c$ is χ-unique.

Proof. Let G be a graph of the form $\theta(3,3,b,b,c,c)$ where $3 \le b \le c$. By Theorem 3, G is χ-unique if $c < 6$. Thus, suppose $c \ge 6$ and $H \sim G$. Let the lowest remaining power and the highest remaining power be denoted by l.r.p. and h.r.p., respectively. By Theorems 6 and 7, we have three cases to consider, that are $H \in g_e(\theta(b_1, b_2, b_3), C_{b_4+1}, C_{b_5+1}, C_{b_6+1})$ where $2 \le b_1 \le b_2 \le b_3$ and $2 \le b_4, b_5, b_6$ (Case A) or $H \in g_e(\theta(b_1, b_2, b_3, b_4), C_{b_5+1}, C_{b_6+1})$ where $2 \le b_1 \le b_2 \le b_3 \le b_4$ and $2 \le b_5, b_6$ (Case B) or $H \in g_e(\theta(b_1, b_2, b_3, b_4, b_5), C_{b_6+1})$ where $2 \le b_1 \le b_2 \le b_3 \le b_4 \le b_5$ and $2 \le b_6$ (Case C).

Case A. $H \in g_e(\theta(b_1, b_2, b_3), C_{b_4+1}, C_{b_5+1}, C_{b_6+1})$ where $2 \le b_1 \le b_2 \le b_3$ and $2 \le b_4, b_5, b_6$. By Theorem 9, we know that $g(G) = g(H) = 6$ and G and H have the same number of shortest cycles. From Theorem 9,

$$2b + 2c + 6 = b_1 + b_2 + b_3 + b_4 + b_5 + b_6, \tag{1}$$

As $G \cong \theta(3,3,b,b,c,c)$ and $H \in g_e(\theta(b_1, b_2, b_3), C_{b_4+1}, C_{b_5+1}, C_{b_6+1})$, we obtain the following after simplification.

$$\begin{aligned} Q(G) = {} & x^{2b+2c+1} + 4x^{2b+c+4} + 2x^{2b+c+3} + x^{2b+7} + 2x^{2b+6} + x^{2b+1} + 4x^{b+2c+4} + 2x^{b+2c+3} + 4x^{b+c+7} \\ & + 8x^{b+c+6} + 4x^{b+c+1} + 2x^{b+9} + 2x^{b+5} + 4x^{b+4} + x^{2c+7} + 2x^{2c+6} + x^{2c+1} + 2x^{c+9} + 2x^{c+5} \\ & + 4x^{c+4} + 2x^8 + x^7 - (x^{2b+2c+2} + 4x^{2b+c+5} + 2x^{2b+c+1} + x^{2b+8} + 3x^{2b+4} + 4x^{b+2c+5} \\ & + 2x^{b+2c+1} + 4x^{b+c+8} + 12x^{b+c+4} + 6x^{b+7} + 2x^{b+1} + x^{2c+8} + 3x^{2c+4} + 6x^{c+7} + 2x^{c+1} \\ & + x^{10} + x^6 + 2x^4) \end{aligned}$$

$$
\begin{aligned}
Q(H) = {} & x^{b_1+b_2+b_3+b_4+b_5} + x^{b_1+b_2+b_3+b_4+b_6} + x^{b_1+b_2+b_3+b_4+1} + x^{b_1+b_2+b_3+b_5+b_6} + x^{b_1+b_2+b_3+b_5+1} + x^{b_1+b_2+b_3+b_6+1} \\
& + x^{b_1+b_2+b_3} + x^{b_1+b_4+b_5+b_6+1} + x^{b_1+b_4+b_5+2} + x^{b_1+b_4+b_6+2} + x^{b_1+b_4+1} + x^{b_1+b_5+b_6+2} + x^{b_1+b_5+1} \\
& + x^{b_1+b_6+1} + x^{b_1+2} + x^{b_2+b_4+b_5+b_6+1} + x^{b_2+b_4+b_5+2} + x^{b_2+b_4+b_6+2} + x^{b_2+b_4+1} + x^{b_2+b_5+b_6+2} + x^{b_2+b_5+1} \\
& + x^{b_2+b_6+1} + x^{b_2+2} + x^{b_3+b_4+b_5+b_6+1} + x^{b_3+b_4+b_5+2} + x^{b_3+b_4+b_6+2} + x^{b_3+b_4+1} + x^{b_3+b_5+b_6+2} + x^{b_3+b_5+1} \\
& + x^{b_3+b_6+1} + x^{b_3+2} + x^{b_4+b_5+b_6+3} + x^{b_4+b_5+1} + x^{b_4+b_6+1} + x^{b_4+3} + x^{b_5+b_6+1} + x^{b_5+3} + x^{b_6+3} - \\
& (x^{b_1+b_2+b_3+b_4+b_5+1} + x^{b_1+b_2+b_3+b_4+b_6+1} + x^{b_1+b_2+b_3+b_4} + x^{b_1+b_2+b_3+b_5+b_6+1} + x^{b_1+b_2+b_3+b_5} + x^{b_1+b_2+b_3+b_6} \\
& + x^{b_1+b_2+b_3+1} + x^{b_1+b_4+b_5+b_6+2} + x^{b_1+b_4+b_5+1} + x^{b_1+b_4+b_6+1} + x^{b_1+b_4+2} + x^{b_1+b_5+b_6+1} + x^{b_1+b_5+2} + x^{b_1+b_6+2} \\
& + x^{b_1+1} + x^{b_2+b_4+b_5+b_6+2} + x^{b_2+b_4+b_5+1} + x^{b_2+b_4+b_6+1} + x^{b_2+b_4+2} + x^{b_2+b_5+b_6+1} + x^{b_2+b_5+2} + x^{b_2+b_6+2} \\
& + x^{b_2+1} + x^{b_3+b_4+b_5+b_6+2} + x^{b_3+b_4+b_5+1} + x^{b_3+b_4+b_6+1} + x^{b_3+b_4+2} + x^{b_3+b_5+b_6+1} + x^{b_3+b_5+2} + x^{b_3+b_6+2} \\
& + x^{b_3+1} + x^{b_4+b_5+b_6+1} + x^{b_4+b_5+3} + x^{b_4+b_6+3} + x^{b_4+1} + x^{b_5+b_6+3} + x^{b_5+1} + x^{b_6+1} + x^{3})
\end{aligned}
$$

Consider the term $-x^3$ in $Q(H)$. This term cannot be cancelled by any positive term in $Q(H)$. There is no term in $Q(G)$ that equal to $-x^3$ as well since $3 \le b \le c$. Thus, we obtain $Q(G) \ne Q(H)$, a contradiction.

Case B. $H \in g_e(\theta(b_1, b_2, b_3, b_4), C_{b_5+1}, C_{b_6+1})$ where $2 \le b_1 \le b_2 \le b_3 \le b_4$ and $2 \le b_5, b_6$. By Theorem 9, we know that *g*(*G*) = *g*(*H*) = 6 and *G* and *H* have the same number of shortest cycles. From Theorem 9,

$$2b + 2c + 6 = b_1 + b_2 + b_3 + b_4 + b_5 + b_6, \qquad (2)$$

As $G \cong$ θ(3,3,*b*,*b*,*c*,*c*) and $H \in g_e(\theta(b_1, b_2, b_3, b_4), C_{b_5+1}, C_{b_6+1})$, we obtain the following after simplification.

$$
\begin{aligned}
Q_1(G) = {} & x^{2b+2c+1} + 4x^{2b+c+4} + 2x^{2b+c+3} + x^{2b+7} + 2x^{2b+6} + x^{2b+1} + 4x^{b+2c+4} + 2x^{b+2c+3} \\
& + 4x^{b+c+7} + 8x^{b+c+6} + 4x^{b+c+1} + 2x^{b+9} + 2x^{b+5} + 4x^{b+4} + x^{2c+7} + 2x^{2c+6} + x^{2c+1} \\
& + 2x^{c+9} + 2x^{c+5} + 4x^{c+4} + 2x^{8} + x^{7} - (x^{2b+2c+2} + 4x^{2b+c+5} + 2x^{2b+c+1} + x^{2b+8} \\
& + 3x^{2b+4} + 4x^{b+2c+5} + 2x^{b+2c+1} + 4x^{b+c+8} + 12x^{b+c+4} + 6x^{b+7} + 2x^{b+1} + x^{2c+8} + 3x^{2c+4} \\
& + 6x^{c+7} + 2x^{c+1} + x^{10} + x^{6} + x^{4})
\end{aligned}
$$

$$
\begin{aligned}
Q_1(H) = {} & x^{b_1+b_2+b_3+b_4+b_5} + x^{b_1+b_2+b_3+b_4+b_6} + x^{b_1+b_2+b_3+b_4+1} + x^{b_1+b_2+b_5+b_6+1} + x^{b_1+b_2+b_5+2} + x^{b_1+b_2+b_6+2} \\
& + x^{b_1+b_2+1} + x^{b_1+b_3+b_5+b_6+1} + x^{b_1+b_3+b_5+2} + x^{b_1+b_3+b_6+2} + x^{b_1+b_3+1} + x^{b_1+b_4+b_5+b_6+1} + x^{b_1+b_4+b_5+2} \\
& + x^{b_1+b_4+b_6+2} + x^{b_1+b_4+1} + x^{b_1+b_5+b_6+3} + x^{b_1+b_5+1} + x^{b_1+b_6+1} + x^{b_1+3} + x^{b_2+b_3+b_5+b_6+1} \\
& + x^{b_2+b_3+b_5+2} + x^{b_2+b_3+b_6+2} + x^{b_2+b_3+1} + x^{b_2+b_4+b_5+b_6+1} + x^{b_2+b_4+b_5+2} + x^{b_2+b_4+b_6+2} + x^{b_2+b_4+1} \\
& + x^{b_2+b_5+b_6+3} + x^{b_2+b_5+1} + x^{b_2+b_6+1} + x^{b_2+3} + x^{b_3+b_4+b_5+b_6+1} + x^{b_3+b_4+b_5+2} + x^{b_3+b_4+b_6+2} \\
& + x^{b_3+b_4+1} + x^{b_3+b_5+b_6+3} + x^{b_3+b_5+1} + x^{b_3+b_6+1} + x^{b_3+3} + x^{b_4+b_5+b_6+3} + x^{b_4+b_5+1} + x^{b_4+b_6+1} \\
& + x^{b_4+3} + x^{b_5+b_6+1} + x^{b_5+4} + x^{b_6+4} - (x^{b_1+b_2+b_3+b_4+b_5+1} + x^{b_1+b_2+b_3+b_4+b_6+1} + x^{b_1+b_2+b_3+b_4} \\
& + x^{b_1+b_2+b_5+b_6+2} + x^{b_1+b_2+b_5+1} + x^{b_1+b_2+b_6+1} + x^{b_1+b_2+2} + x^{b_1+b_3+b_5+b_6+2} + x^{b_1+b_3+b_5+1} \\
& + x^{b_1+b_3+b_6+1} + x^{b_1+b_3+2} + x^{b_1+b_4+b_5+b_6+2} + x^{b_1+b_4+b_5+1} + x^{b_1+b_4+b_6+1} + x^{b_1+b_4+2} + x^{b_1+b_5+b_6+1} \\
& + x^{b_1+b_5+3} + x^{b_1+b_6+3} + x^{b_1+1} + x^{b_2+b_3+b_5+b_6+2} + x^{b_2+b_3+b_5+1} + x^{b_2+b_3+b_6+1} + x^{b_2+b_3+2} \\
& + x^{b_2+b_4+b_5+b_6+2} + x^{b_2+b_4+b_5+1} + x^{b_2+b_4+b_6+1} + x^{b_2+b_4+2} + x^{b_2+b_5+b_6+1} + x^{b_2+b_5+3} + x^{b_2+b_6+3} \\
& + x^{b_2+1} + x^{b_3+b_4+b_5+b_6+2} + x^{b_3+b_4+b_5+1} + x^{b_3+b_4+b_6+1} + x^{b_3+b_4+2} + x^{b_3+b_5+b_6+1} + x^{b_3+b_5+3} \\
& + x^{b_3+b_6+3} + x^{b_3+1} + x^{b_4+b_5+b_6+1} + x^{b_4+b_5+3} + x^{b_4+b_6+3} + x^{b_4+1} + x^{b_5+b_6+4} + x^{b_5+1} + x^{b_6+1})
\end{aligned}
$$

Without loss of generality, we have three cases to consider.

Case 1. $b_5 = b_6 = 5$. Now, H has two cycles of length 6. But, we know that H has only one cycle of length 6, a contradiction.

Case 2. $b_5 = 5$, $b_6 \neq 5$. By Equation (2), we obtain

$$2b + 2c + 1 = b_1 + b_2 + b_3 + b_4 + b_6, \tag{3}$$

Simplifying $Q_1(G)$ and $Q_1(H)$, we obtain,

$$
\begin{aligned}
Q_2(G) = {} & 4x^{2b+c+4} + 2x^{2b+c+3} + x^{2b+7} + 2x^{2b+6} + x^{2b+1} + 4x^{b+2c+4} + 2x^{b+2c+3} + 4x^{b+c+7} + 8x^{b+c+6} \\
& + 4x^{b+c+1} + 2x^{b+9} + 2x^{b+5} + 4x^{b+4} + x^{2c+7} + 2x^{2c+6} + x^{2c+1} + 2x^{c+9} + 2x^{c+5} + 4x^{c+4} + 2x^{8} \\
& + x^{7} - (4x^{2b+c+5} + 2x^{2b+c+1} + x^{2b+8} + 3x^{2b+4} + 4x^{b+2c+5} + 2x^{b+2c+1} + 4x^{b+c+8} + 12x^{b+c+4} \\
& + 6x^{b+7} + 2x^{b+1} + x^{2c+8} + 3x^{2c+4} + 6x^{c+7} + 2x^{c+1} + x^{10} + x^{4})
\end{aligned}
$$

$$
\begin{aligned}
Q_2(H) = {} & x^{b_1+b_2+b_3+b_4+5} + x^{b_1+b_2+b_3+b_4+1} + x^{b_1+b_2+b_6+6} + x^{b_1+b_2+b_6+2} + x^{b_1+b_2+7} + x^{b_1+b_2+1} + x^{b_1+b_3+b_6+6} \\
& + x^{b_1+b_3+b_6+2} + x^{b_1+b_3+7} + x^{b_1+b_3+1} + x^{b_1+b_4+b_6+6} + x^{b_1+b_4+b_6+2} + x^{b_1+b_4+7} + x^{b_1+b_4+1} + x^{b_1+b_6+8} \\
& + x^{b_1+b_6+1} + x^{b_1+6} + x^{b_1+3} + x^{b_2+b_3+b_6+6} + x^{b_2+b_3+b_6+2} + x^{b_2+b_3+7} + x^{b_2+b_3+1} + x^{b_2+b_4+b_6+6} \\
& + x^{b_2+b_4+b_6+2} + x^{b_2+b_4+7} + x^{b_2+b_4+1} + x^{b_2+b_6+8} + x^{b_2+b_6+1} + x^{b_2+6} + x^{b_2+3} + x^{b_3+b_4+b_6+6} \\
& + x^{b_3+b_4+b_6+2} + x^{b_3+b_4+7} + x^{b_3+b_4+1} + x^{b_3+b_6+8} + x^{b_3+b_6+1} + x^{b_3+6} + x^{b_3+3} + x^{b_4+b_6+8} + x^{b_4+b_6+1} \\
& + x^{b_4+6} + x^{b_4+3} + x^{b_6+6} + x^{b_6+4} + x^{9} - \big(x^{b_1+b_2+b_3+b_4+6} + x^{b_1+b_2+b_3+b_4} + x^{b_1+b_2+b_6+7} + x^{b_1+b_2+b_6+1} \\
& + x^{b_1+b_2+6} + x^{b_1+b_2+2} + x^{b_1+b_3+b_6+7} + x^{b_1+b_3+b_6+1} + x^{b_1+b_3+6} + x^{b_1+b_3+2} + x^{b_1+b_4+b_6+7} + x^{b_1+b_4+b_6+1} \\
& + x^{b_1+b_4+6} + x^{b_1+b_4+2} + x^{b_1+b_6+6} + x^{b_1+b_6+3} + x^{b_1+8} + x^{b_1+1} + x^{b_2+b_3+b_6+7} + x^{b_2+b_3+b_6+1} + x^{b_2+b_3+6} \\
& + x^{b_2+b_3+2} + x^{b_2+b_4+b_6+7} + x^{b_2+b_4+b_6+1} + x^{b_2+b_4+6} + x^{b_2+b_4+2} + x^{b_2+b_6+6} + x^{b_2+b_6+3} + x^{b_2+8} + x^{b_2+1} \\
& + x^{b_3+b_4+b_6+7} + x^{b_3+b_4+b_6+1} + x^{b_3+b_4+6} + x^{b_3+b_4+2} + x^{b_3+b_6+6} + x^{b_3+b_6+3} + x^{b_3+8} + x^{b_3+1} + x^{b_4+b_6+6} \\
& + x^{b_4+b_6+3} + x^{b_4+8} + x^{b_4+1} + x^{b_6+9} + x^{b_6+1}\big)
\end{aligned}
$$

Compare the l.r.p. in $Q_2(G)$ and the l.r.p. in $Q_2(H)$. We know that $g(H) = 6$, thus $b_6 \geq 6$. Then we obtain $b_1 = 3$ or $b_2 = 3$ or $b_3 = 3$ or $b_4 = 3$.

Case 2.1. $b_1 = 3$. We obtain the following after simplification.

$$
\begin{aligned}
Q_3(G) = {} & 4x^{2b+c+4} + 2x^{2b+c+3} + x^{2b+7} + 2x^{2b+6} + x^{2b+1} + 4x^{b+2c+4} + 2x^{b+2c+3} + 4x^{b+c+7} + 8x^{b+c+6} \\
& + 4x^{b+c+1} + 2x^{b+9} + 2x^{b+5} + 4x^{b+4} + x^{2c+7} + 2x^{2c+6} + x^{2c+1} + 2x^{c+9} + 2x^{c+5} + 4x^{c+4} \\
& + 2x^{8} + x^{7} - \big(4x^{2b+c+5} + 2x^{2b+c+1} + x^{2b+8} + 3x^{2b+4} + 4x^{b+2c+5} + 2x^{b+2c+1} + 4x^{b+c+8} \\
& + 12x^{b+c+4} + 6x^{b+7} + 2x^{b+1} + x^{2c+8} + 3x^{2c+4} + 6x^{c+7} + 2x^{c+1} + x^{10}\big)
\end{aligned}
$$

$$
\begin{aligned}
Q_3(H) = {} & x^{b_2+b_3+b_4+8} + x^{b_2+b_3+b_4+4} + x^{b_2+b_3+b_6+6} + x^{b_2+b_3+b_6+2} + x^{b_2+b_3+7} + x^{b_2+b_3+1} + x^{b_2+b_4+b_6+6} \\
& + x^{b_2+b_4+b_6+2} + x^{b_2+b_4+7} + x^{b_2+b_4+1} + x^{b_2+b_6+9} + x^{b_2+b_6+8} + x^{b_2+b_6+5} + x^{b_2+b_6+1} + x^{b_2+10} \\
& + x^{b_2+6} + x^{b_2+4} + x^{b_2+3} + x^{b_3+b_4+b_6+6} + x^{b_3+b_4+b_6+2} + x^{b_3+b_4+7} + x^{b_3+b_4+1} + x^{b_3+b_6+9} \\
& + x^{b_3+b_6+8} + x^{b_3+b_6+5} + x^{b_3+b_6+1} + x^{b_3+10} + x^{b_3+6} + x^{b_3+4} + x^{b_3+3} + x^{b_4+b_6+9} + x^{b_4+b_6+8} \\
& + x^{b_4+b_6+5} + x^{b_4+b_6+1} + x^{b_4+10} + x^{b_4+6} + x^{b_4+4} + x^{b_4+3} + x^{b_6+11} + 2x^{b_6+4} + 2x^{9} + x^{6} \\
& - \big(x^{b_2+b_3+b_4+9} + x^{b_2+b_3+b_4+3} + x^{b_2+b_3+b_6+7} + x^{b_2+b_3+b_6+1} + x^{b_2+b_3+6} + x^{b_2+b_3+2} \\
& + x^{b_2+b_4+b_6+7} + x^{b_2+b_4+b_6+1} + x^{b_2+b_4+6} + x^{b_2+b_4+2} + x^{b_2+b_6+10} + x^{b_2+b_6+6} + x^{b_2+b_6+4} \\
& + x^{b_2+b_6+3} + x^{b_2+9} + x^{b_2+8} + x^{b_2+5} + x^{b_2+1} + x^{b_3+b_4+b_6+7} + x^{b_3+b_4+b_6+1} + x^{b_3+b_4+6} \\
& + x^{b_3+b_4+2} + x^{b_3+b_6+10} + x^{b_3+b_6+6} + x^{b_3+b_6+4} + x^{b_3+b_6+3} + x^{b_3+9} + x^{b_3+8} + x^{b_3+5} + x^{b_3+1} \\
& + x^{b_4+b_6+10} + x^{b_4+b_6+6} + x^{b_4+b_6+4} + x^{b_4+b_6+3} + x^{b_4+9} + x^{b_4+8} + x^{b_4+5} + x^{b_4+1} + 2x^{b_6+9} \\
& + x^{b_6+1} + x^{11}\big)
\end{aligned}
$$

Consider the l.r.p. in $Q_3(G)$ and the l.r.p. in $Q_3(H)$. We have $b_2 = 5$ or $b_3 = 5$ or $b_4 = 5$.

Case 2.1.1. $b_2 = 5$. We obtain the following after simplification.

$$\begin{aligned}Q_4(G) = {} & 4x^{2b+c+4} + 2x^{2b+c+3} + x^{2b+7} + 2x^{2b+6} + x^{2b+1} + 4x^{b+2c+4} + 2x^{b+2c+3} + 4x^{b+c+7} \\ & + 8x^{b+c+6} + 4x^{b+c+1} + 2x^{b+9} + 2x^{b+5} + 4x^{b+4} + x^{2c+7} + 2x^{2c+6} + x^{2c+1} + 2x^{c+9} + 2x^{c+5} \\ & + 4x^{c+4} + x^8 + x^7 - (4x^{2b+c+5} + 2x^{2b+c+1} + x^{2b+8} + 3x^{2b+4} + 4x^{b+2c+5} + 2x^{b+2c+1} \\ & + 4x^{b+c+8} + 12x^{b+c+4} + 6x^{b+7} + 2x^{b+1} + x^{2c+8} + 3x^{2c+4} + 6x^{c+7} + 2x^{c+1})\end{aligned}$$

$$\begin{aligned}Q_4(H) = {} & x^{b_3+b_4+b_6+6} + x^{b_3+b_4+b_6+2} + x^{b_3+b_4+13} + x^{b_3+b_4+9} + x^{b_3+b_4+7} + x^{b_3+b_4+1} + x^{b_3+b_6+11} + x^{b_3+b_6+9} \\ & + x^{b_3+b_6+8} + x^{b_3+b_6+7} + x^{b_3+b_6+5} + x^{b_3+b_6+1} + x^{b_3+12} + x^{b_3+10} + 2x^{b_3+6} + x^{b_3+4} + x^{b_3+3} \\ & + x^{b_4+b_6+11} + x^{b_4+b_6+9} + x^{b_4+b_6+8} + x^{b_4+b_6+7} + x^{b_4+b_6+5} + x^{b_4+b_6+1} + x^{b_4+12} + x^{b_4+10} + 2x^{b_4+6} \\ & + x^{b_4+4} + x^{b_4+3} + x^{b_6+14} + x^{b_6+13} + x^{b_6+10} + x^{b_6+6} + 2x^{b_6+4} + x^{15} + 3x^9 - (x^{b_3+b_4+b_6+7} \\ & + x^{b_3+b_4+b_6+1} + x^{b_3+b_4+14} + x^{b_3+b_4+8} + x^{b_3+b_4+6} + x^{b_3+b_4+2} + x^{b_3+b_6+12} + x^{b_3+b_6+10} + 2x^{b_3+b_6+6} \\ & + x^{b_3+b_6+4} + x^{b_3+b_6+3} + x^{b_3+11} + x^{b_3+9} + x^{b_3+8} + x^{b_3+7} + x^{b_3+5} + x^{b_3+1} + x^{b_4+b_6+12} + x^{b_4+b_6+10} \\ & + 2x^{b_4+b_6+6} + x^{b_4+b_6+4} + x^{b_4+b_6+3} + x^{b_4+11} + x^{b_4+9} + x^{b_4+8} + x^{b_4+7} + x^{b_4+5} + x^{b_4+1} + x^{b_6+15} \\ & + 3x^{b_6+9} + x^{b_6+8} + x^{b_6+1} + x^{14} + x^{13})\end{aligned}$$

Considering the l.r.p. in $Q_4(G)$, we have $b = 6$ or $c = 6$.

Case 2.1.1.1. $b = 6$. Since the term $-x^{b+1}$ has coefficient 2, then there is one term in $Q_4(H)$ that equal to $-x^7$. Then we have $b_3 = 6$ or $b_4 = 6$ or $b_6 = 6$.

Case 2.1.1.1(a). If $b_3 = 6$, by Equation (3), $2c = b_4+b_6+1$. We obtain $b_4 = 7$ or $b_6 = 7$. If $b_4 = 7$, then we have $Q_4(G) \neq Q_4(H)$, a contradiction. If $b_6 = 7$, then $Q_4(G) \neq Q_4(H)$, a contradiction.

Case 2.1.1.1(b). If $b_4 = 6$, similar to Case 2.1.1.1(a), we obtain a contradiction.

Case 2.1.1.1(c). If $b_6 = 6$, similar to Case 2.1.1.1(a), we obtain a contradiction.

Case 2.1.1.2. $c = 6$. For the same reason as in Case 2.1.1.1, we have $b_3 = 6$ or $b_4 = 6$ or $b_6 = 6$.

Case 2.1.1.2(a) . If $b_3 = 6$, by Equation (3), $2b = b_4+b_6+1$. Then, we obtain $Q_4(G) \neq Q_4(H)$ after simplification, a contradiction.

Case 2.1.1.2(b). If $b_4 = 6$, similar to Case 2.1.1.2(a), we obtain a contradiction.

Case 2.1.1.2(c). If $b_6 = 6$, similar to Case 2.1.1.2(a), we obtain a contradiction.

Case 2.1.2. $b_3 = 5$. Then we obtain the following after simplification.

$$
\begin{aligned}
Q_5(G) = {} & 4x^{2b+c+4} + 2x^{2b+c+3} + x^{2b+7} + 2x^{2b+6} + x^{2b+1} + 4x^{b+2c+4} + 2x^{b+2c+3} + 4x^{b+c+7} + 8x^{b+c+6} \\
& + 4x^{b+c+1} + 2x^{b+9} + 2x^{b+5} + 4x^{b+4} + x^{2c+7} + 2x^{2c+6} + x^{2c+1} + 2x^{c+9} + 2x^{c+5} + 4x^{c+4} + x^{8} \\
& + x^{7} - (4x^{2b+c+5} + 2x^{2b+c+1} + x^{2b+8} + 3x^{2b+4} + 4x^{b+2c+5} + 2x^{b+2c+1} + 4x^{b+c+8} + 12x^{b+c+4} \\
& + 6x^{b+7} + 2x^{b+1} + x^{2c+8} + 3x^{2c+4} + 6x^{c+7} + 2x^{c+1})
\end{aligned}
$$

$$
\begin{aligned}
Q_5(H) = {} & x^{b_2+b_4+b_6+6} + x^{b_2+b_4+b_6+2} + x^{b_2+b_4+13} + x^{b_2+b_4+9} + x^{b_2+b_4+7} + x^{b_2+b_4+1} + x^{b_2+b_6+11} + x^{b_2+b_6+9} \\
& + x^{b_2+b_6+8} + x^{b_2+b_6+7} + x^{b_2+b_6+5} + x^{b_2+b_6+1} + x^{b_2+12} + x^{b_2+10} + 2x^{b_2+6} + x^{b_2+4} + x^{b_2+3} + x^{b_4+b_6+11} \\
& + x^{b_4+b_6+9} + x^{b_4+b_6+8} + x^{b_4+b_6+7} + x^{b_4+b_6+5} + x^{b_4+b_6+1} + x^{b_4+12} + x^{b_4+10} + 2x^{b_4+6} + x^{b_4+4} + x^{b_4+3} \\
& + x^{b_6+14} + x^{b_6+13} + x^{b_6+10} + x^{b_6+6} + 2x^{b_6+4} + x^{15} + 3x^{9} - (x^{b_2+b_4+b_6+7} + x^{b_2+b_4+b_6+1} + x^{b_2+b_4+14} \\
& + x^{b_2+b_4+8} + x^{b_2+b_4+6} + x^{b_2+b_4+2} + x^{b_2+b_6+12} + x^{b_2+b_6+10} + 2x^{b_2+b_6+6} + x^{b_2+b_6+4} + x^{b_2+b_6+3} + x^{b_2+11} \\
& + x^{b_2+9} + x^{b_2+8} + x^{b_2+7} + x^{b_2+5} + x^{b_2+1} + x^{b_4+b_6+12} + x^{b_4+b_6+10} + 2x^{b_4+b_6+6} + x^{b_4+b_6+4} + x^{b_4+b_6+3} \\
& + x^{b_4+11} + x^{b_4+9} + x^{b_4+8} + x^{b_4+7} + x^{b_4+5} + x^{b_4+1} + x^{b_6+15} + 3x^{b_6+9} + x^{b_6+8} + x^{b_6+1} + x^{14} + x^{13})
\end{aligned}
$$

Consider the l.r.p. in $Q_5(G)$, we have $b = 6$ or $c = 6$.

Case 2.1.2.1. $b = 6$. For the same reason as in Case 2.1.1.1, we have $b_4 = 6$ or $b_6 = 6$.

Case 2.1.2.1(a) . $b_4 = 6$. By Equation (3), $2c = b_2+b_6+1$. Simplifying $Q_5(G)$ and $Q_5(H)$, we obtain $c = 7$ and $b_2 = 3$ or $b_6 = 7$.

If $b_2 = 3$, we obtain $b_6 = 10$ and $Q_5(G) \neq Q_5(H)$, a contradiction. If $b_6 = 7$, we obtain $b_2 = 6$, a contradiction since $3 \leq b_2 \leq 5$.

Case 2.1.2.1(b). b_6 = 6. By Equation (3), $2c = b_2+b_4+1$. Simplifying $Q_5(G)$ and $Q_5(H)$, we obtain c = 7 and b_2 = 3 or b_6 = 7. Similar to Case 2.1.2.1(a), we obtain a contradiction.

Case 2.1.2.2. c = 6. For the same reason as in Case 2.1.1.1, we have b_4 = 6 or b_6 = 6.

Case 2.1.2.2(a). b_4 = 6. By Equation (3), $2c = b_2+b_6+1$. Simplifying $Q_5(G)$ and $Q_5(H)$, we obtain b = 5 and b_2 = 5. Then, we have $Q_5(G) \neq Q_5(H)$, a contradiction.

Case 2.1.2.2(b). b_6 = 6. Similar to Case 2.1.2.2(a), we obtain a contradiction.

Case 2.1.3. b_4 = 5. Simplifying $Q_3(G)$ and $Q_3(H)$, we obtain b = 6 or c = 6. Similar to the proofs of Cases 2.1.1 and 2.1.2, we obtain a contradiction for each subcases.

Case 2.2. b_2 = 3. By using similar method as in Case 2.1, we obtain a contradiction.

Case 2.3. b_3 = 3. By using similar method as in Case 2.1, we obtain a contradiction.

Case 2.4. b_4 = 3. By using similar method as in Case 2.1, we obtain a contradiction.

Case 3. $b_5 \neq 5$, $b_6 \neq 5$. We can prove this case using the similar method as in Cases 1 and 2.

Case C. $H \in g_e(\theta(b_1,b_2,b_3,b_4,b_5),C_{b_6+1})$ where $2 \le b_1 \le b_2 \le b_3 \le b_4 \le b_5$ and $2 \le b_6$. Using a similar method as in Cases A and B, we obtain that G is χ-unique.

This completes the proof. □

Remark. For the detailed proof of Theorem 11, please refer to Karim, 2015.

4. Conclusion

In this paper, we have investigated the chromaticity of $\theta(3,3,b,b,c,c)$ where $3 \le b \le c$. Therefore, it is natural to consider the general case of such graph. In the next study, we will consider the chromaticity of the general case of 6-bridge graph $\theta\,(a,a,b,b,c,c)$ where $2 \le a \le b \le c$.

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